On lines in a triangle tangent to a conic

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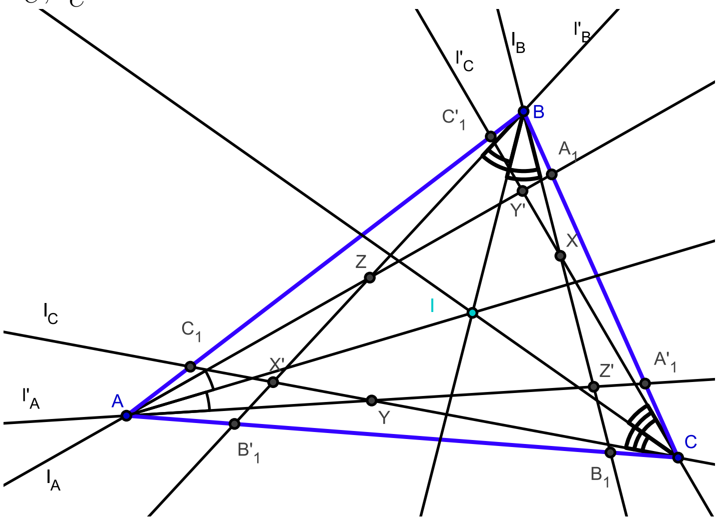
Abstract. We present generalizations of theorems on *Kypert's construction* and on *2nd Morley's Centre*. Most of our proofs are synthetic.

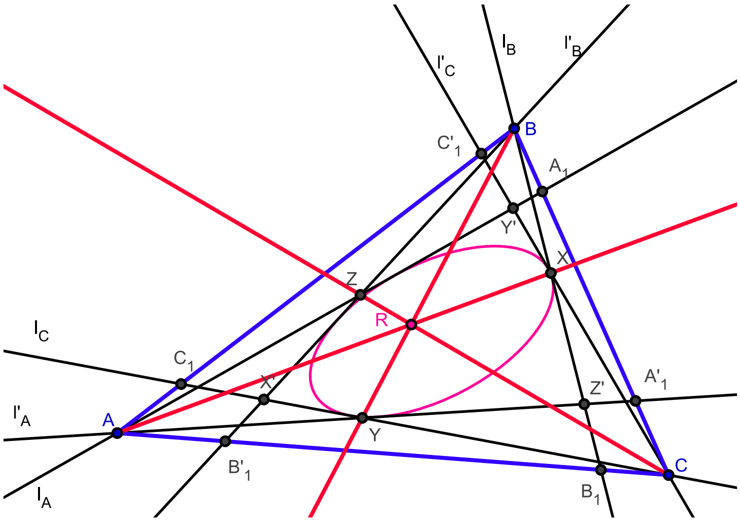
Introduction and main results

Let us introduce necessary notation.

Let us fix $\triangle ABC$. Draw lines l_A, l_B and l_C through vertices A, B and C, respectively. Lines l'_A, l'_B and l'_C are symmetric to l_A, l_B and l_C with respect to bisecting lines of $\triangle ABC$. Let X, Y and Z be the intersection points of l_B and l'_C, l_C and l'_A, l_A and l'_B, respectively. Let X', Y' and Z' be the intersection points of l'_B and l_C, l'_C and l_A, l'_A and l_B, respectively. Let A_1, A'_1, B_1, B'_1, C_1 and C'_1 be the intersection points of lines l_A, l'_A, l_B, l'_B and l_C, l'_C with sidelines of $\triangle ABC$.

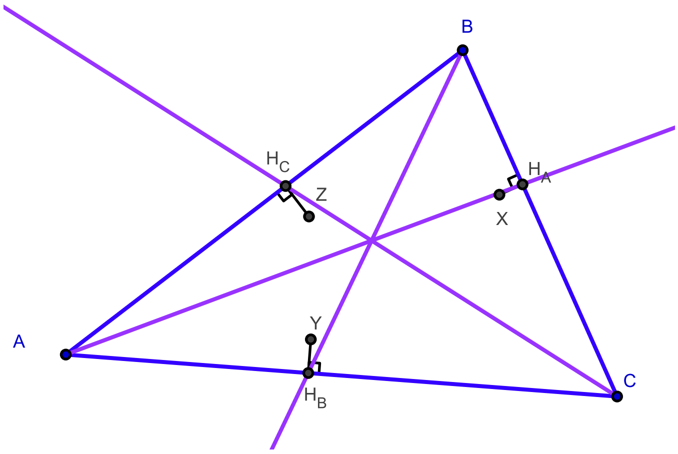


Theorem 1. *Lines AX, BY and CZ intersect at a point.*



Corollary. *Lines l_A, l'_A, l_B, l'_B, l_C and l'_C are tangent to a conic.*

Theorem 2. *Let H_A be the intersection point of XH_A and the line perpendicular to BC and passing through X. Define H_B and H_C analogously. Then lines AH_A, BH_B and CH_C intersect at one point.*

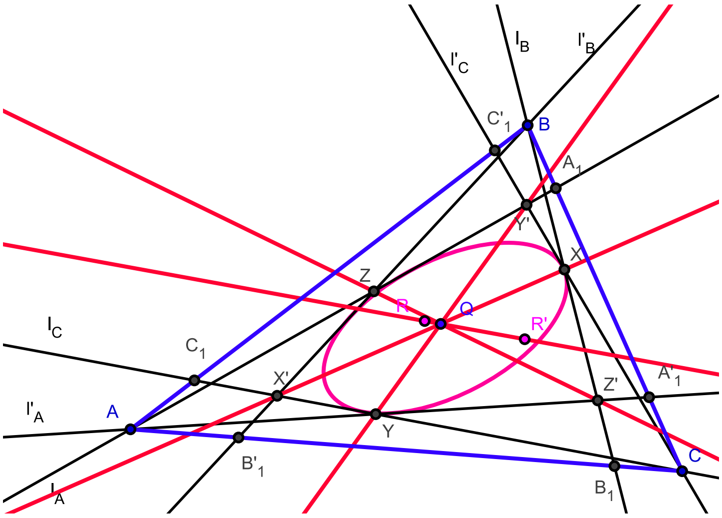


Remark. Points H'_A, H'_B, H'_C are defined similarly to points H_A, H_B and H_C. Then points H_A, H_B,H_C, H'_A, H'_B and H'_C lie on the same conic section. This follows by a criterion for being conconic (see §2), theorem 2 and Ceva's theorem.

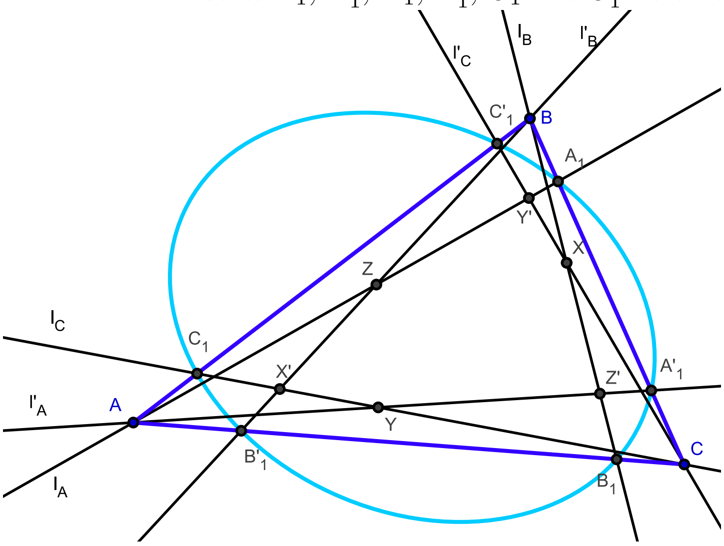
Theorem 3. *Let R' be the intersection point of AX', BY' and CZ' (these lines intersect at a point by Theorem 1). Lines XX', YY', ZZ' and RR' are intersect at a point.*

Remark. Clearly, X' is the isogonal conjugate of X, Y' is the isogonal conjugate of Y and Z' is the isogonal conjugate of Z'. This means that R' is the isogonal conjugate of R.

Denote by Q the intersection point of XX', YY', ZZ' and RR'.



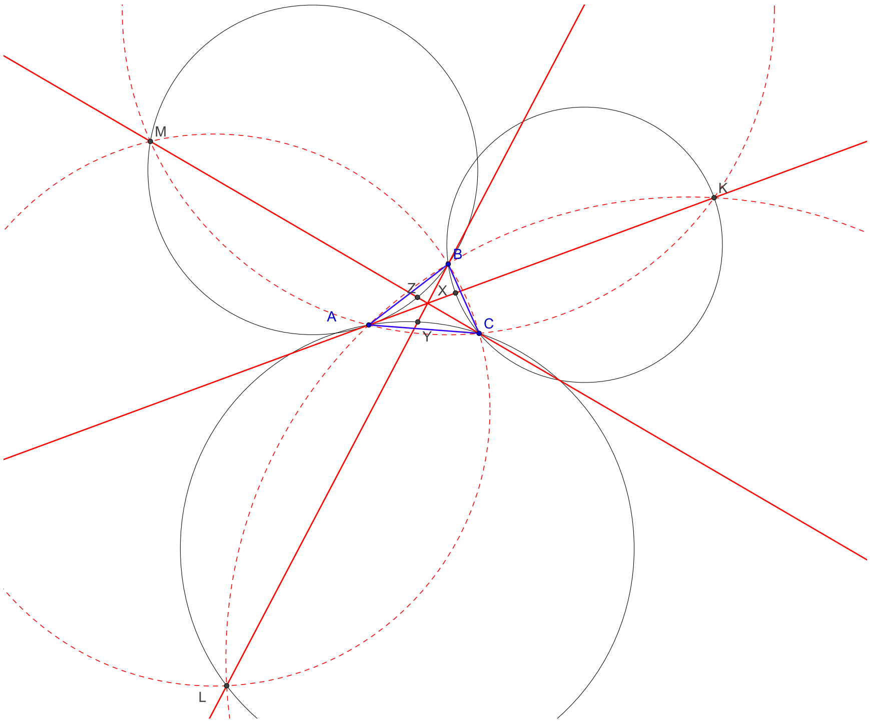
Theorem 4. *Points A_1, A_1', B_1, B_1', C_1 and C_1' lie on the same conic.*



A simple version of theorems is presented at the appendix.

Proofs and some new corollaries

Proof of Theorem 1. Cycles $\omega(\triangle XBC)$, $\omega(\triangle YAC)$ and $\omega(\triangle ZAB)$ are drawn through $\triangle XBC$, $\triangle YAC$ and $\triangle ZAB$. Point K is second point of intersection AX and $\omega(\triangle XBC)$. Points L and M are similarly defined (for BY and CZ accordingly).



$$\angle CMB = \angle ZAB = \angle YAC = \angle BLC$$

We have that points M, L, B and C are concyclic[1]. Let's define this cycle ω_x. Cycles ω_y and ω_z are similarly defined. It's clear that lines AX, BY and CZ are radical axises this cycles. This mean that R (point of intersection lines AX, BY and CZ) is radical centre.$\triangleleft$

Proof of Corollary. Let's fix lines l_A, l_B, l_C, l'_A, l'_B. Consider conic, which this five lines tangent it (certainly, we can find the only one conic, which five lines general provision tangent this conic - [1]). This conic is inscribed in to $\angle ZAZ'$ and in to $\angle XBX'$. It means that point P' (points P and P' are two focuses of this conic) is isogonal conjugate of P[2]. Consequently, tangent lines, which are drawn through vertice C, are isogonal lines.

Proposition: Let us fix $\triangle ABC$ and a conic. Tangent lines d_A, d'_A, d_B, d'_B, d_C and d'_C (this lines were drawn through vertices A, B and C, respectively) to conic create hexagon $XY'ZX'YZ'$. Lines AX, BY, CZ are concur. This proposition is conclusion of Brianchon's theorem[3] in next order: d_A, d'_A, d_B, d'_B, d_C, d'_C.

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[1]This formula is true only for this picture (it means that arrangement of lines l_A, l_B and l_C can be very different), but in that cases we'll proof that points M, L, B and C are concyclic in a similar way

[2]Here I used isogonal properties of conics - [1]

[3]I used here next formulation of this theorem: Lines l_i, $i = 1, ..., 6$ tangent the same conic section. Point A_{ij} is point of intersection lines l_i, l_j. Lines $A_{12}A_{45}$, $A_{23}A_{56}$ and $A_{34}A_{61}$ are concur then.

Proof of Theorem 2.

$$\frac{XH_A}{BH_A} = tg(\angle(BX, BC)) \Rightarrow \frac{BH_A}{CH_A} = \frac{tg(\angle(XC, BC))}{tg(\angle(BX, BC))}.$$

We can take that:

$$\frac{CH_B}{AH_B} = \frac{tg(\angle(YA, AC))}{tg(\angle(YC, AC))} = \frac{tg(\angle(AY, AC))}{tg(\angle(XC, BC))};$$

$$\frac{AH_C}{BH_C} = \frac{tg(\angle(BZ, AB))}{tg(\angle(AZ, AB))} = \frac{tg(\angle(BX, BC))}{tg(\angle(AY, AC))} \Rightarrow$$

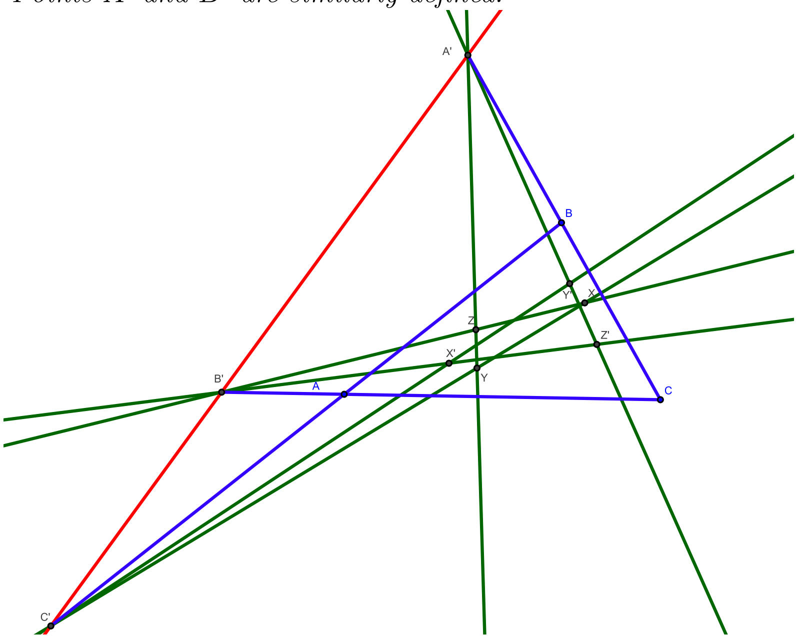
$$\frac{AH_C}{BH_C} \cdot \frac{BH_A}{CH_A} \cdot \frac{CH_B}{AH_B} = 1 \Rightarrow$$

This formula shows that three lines are concur (there is converse proposition of Ceva's theorem).$\triangleleft$

Proof of Theorem 1. Let's proof general result: Let us fix $\triangle ABC$ and a conic. Tangent lines d_A, d'_A, d_B, d'_B, d_C and d'_C (this lines were drawn through vertices A, B and C, respectively) to conic create hexagon $XY'ZX'YZ'$. Points R and R' are defined like before. Then lines XX', YY', ZZ' and RR' are concur.

Let's proof next lemma:

Lemma. *Lines XY,$X'Y'$ and AB are concur. Let's define this point C'. Points A' and B' are similarly defined.*



Proof.: Let's look at $\triangle X'YB$ and $\triangle XY'A$: $X'Y$ cross with XY' at a vertice C; BY cross with AX at a point R and $X'B$ cross with $Y'A$ at a point Z. We have that points C, R and Z are collinear. It means lines

XY,$X'Y'$ and AB are concur (there is converse proposition of Desargues'[4] theorem).

Let's return to the proof of this theorem: we have that $\triangle XYZ$ and $\triangle ABC$ are perspective. This means that points A', B' and C' are collinear (the perspectrix). Now we can conclude that $\triangle XYZ$ and $\triangle X'Y'Z'$ are perspective. It means that lines XX', YY' and ZZ' are concur (but we can proof this propositision with Brianchon's theorem). Q is the point of intersection XX', YY' and ZZ'. Let's look at $\triangle RYZ$ and $\triangle R'Y'Z'$: RY cross with $R'Y'$ at a vertice B, RZ cross with $R'Z'$ at a vertice C and YZ cross with $Y'Z'$ at a point A' (in lemma we proofed that A' lays on BC). It means that $\triangle RYZ$ and $\triangle R'Y'Z'$ are perspective (there is converse proposition of Desargues' theorem). Now we have that point Q lays on line RR'.$\triangleleft$

Proof of Theorem 4. This theorem is equivalent to the converse preposition which we generalized: Points A_1, A'_1, B_1, B'_1, C_1 and C'_1 lie on sidelines BC, AC and AB accordingly. Lines d_A, d'_A, d_B, d'_B, d_C and l'_C tangent the one conic if and only if A_1, A'_1, B_1, B'_1, C_1 and C'_1 lie on the same conic.

I used next proposition (task 14 in [1]) - criterion of conconic:

Let's fix $\triangle ABC$. Points A_1, A_2 lie on sideline BC, B_1, B_2 lie on sideline AC and C_1, C_2 lie on sideline AB. This six points are conconic if and only if, when

$$\frac{BA_1 \cdot BA_2}{CA_1 \cdot CA_2} \cdot \frac{CB_1 \cdot CB_2}{AB_1 \cdot AB_2} \cdot \frac{AC_1 \cdot AC_2}{BC_1 \cdot BC_2} = 1$$

Let K, M, L be intersection points of lines AX with BC, BY with AC, CZ with AB accordingly. We can take (Ceva's equality):

$$\frac{CK}{BK}\frac{BC_2}{AC_2}\frac{AB_1}{CB_1} = 1; \frac{AM}{CM}\frac{CA_2}{BA_2}\frac{BC_1}{AC_1} = 1; \frac{BL}{AL}\frac{AB_2}{CB_2}\frac{CA_1}{BA_1} = 1$$

Let's multiply this three equalities and let's use criterion of conconic:

$$\frac{CK}{BK}\frac{AM}{CM}\frac{BL}{AL} = 1$$

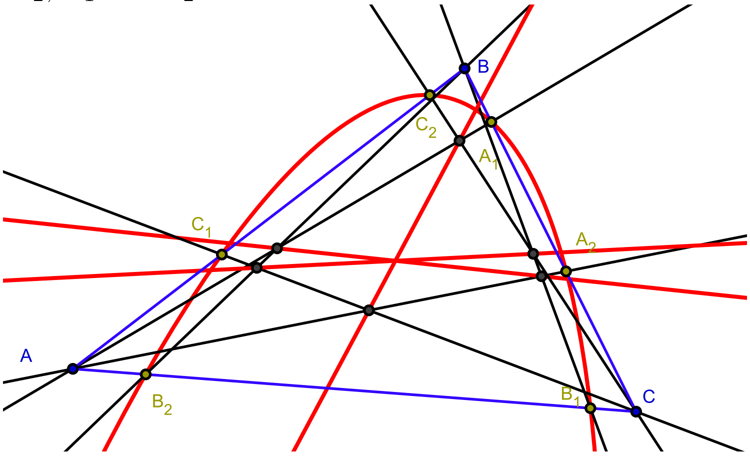
We have that lines AX, BY and CZ are concur (there is converse proposition of Ceva's theorem). Now, the proposition of this theorem is obvious (there is converse proposition of Brianchon's theorem or we can thinking like in theorem 1).$\triangleleft$

Corollary 1. Let A_2 be the intersection point of lines C_1B_1 and $C'_1B'_1$. Points B_2 and C_2 are similarly defined. We can have that point A_2 lie on line XX' (Pappus's[5] theorem). We can receive next result after it:

[4]If the three straight lines joining the corresponding vertices of two triangles $\triangle ABC$ and $\triangle A'B'C'$ all meet in a point (the perspector), then the three intersections of pairs of corresponding sides lie on a straight line (the perspectrix).

[5]Points A_1, B_1, C_1 are collinear and points A_2, B_2, C_2 are collinear too. Then three intersection points of lines A_1B_2 and A_2B_1, B_1C_2 and B_2C_1, C_1A_2 and C_2A_1 are incident to a (third) straight line (next Pappus's line).

Points A_1, A_2, B_1, B_2, C_1 and C_2 lie on sidelines BC, AC and AB accordingly. Three Pappus's lines are concur if and only if points A_1, A_2, B_1, B_2, C_1 and C_2 are conconic.



Corollary 2. Let A_3 be the intersection point of A_1C_1' and B_1A_1'. Points B_3 and C_3 are similarly defined. Then lines AA_3, BB_3 and CC_3 are concur.

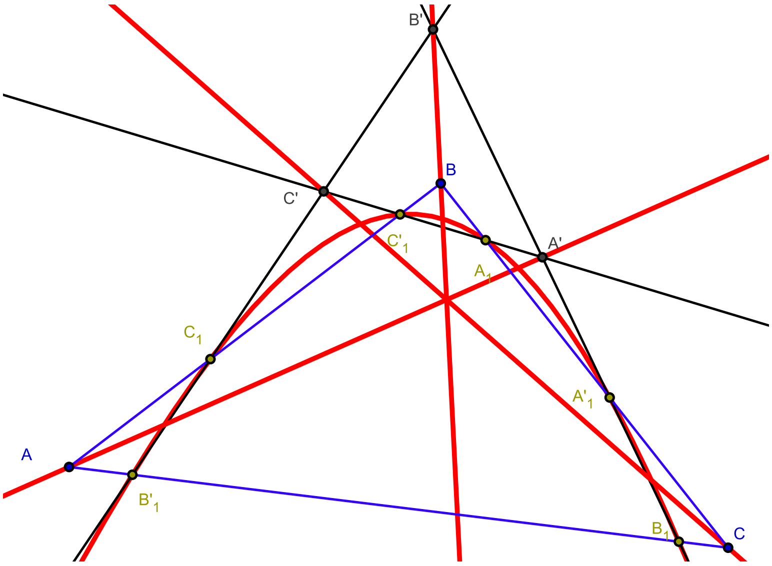
Proof.: Pascal's[6] theorem says that three points of intersection lines AB with $A_1'B_1$, BC with $B_1'C_1$ and AC with A_1C_1' are collinear. It means that $\triangle ABC$ and $\triangle A_3B_3C_3$ are perspective (there is converse proposition of Desargues' theorem).$\triangleleft$

Remark. We can this result generalize: Let's look at two triangles: $\triangle ABC$ and $\triangle A'B'C'$. Points C_1 and B_1' are points of intersection lines $C'B'$ with AB and AC accordingly. Points A_1, A_1', B_1, C_1' are similarly defined. This two triangles are perspective if and only if points A_1, A_1', B_1, B_1', C_1 and C_1' are conconic[7].

[6]The theorem states that if a hexagon is inscribed in a conic, then the three points at which the pairs of opposite sides meet, lie on a straight line.

It is obvious that this theorem is generalization of Pappus's theorem.

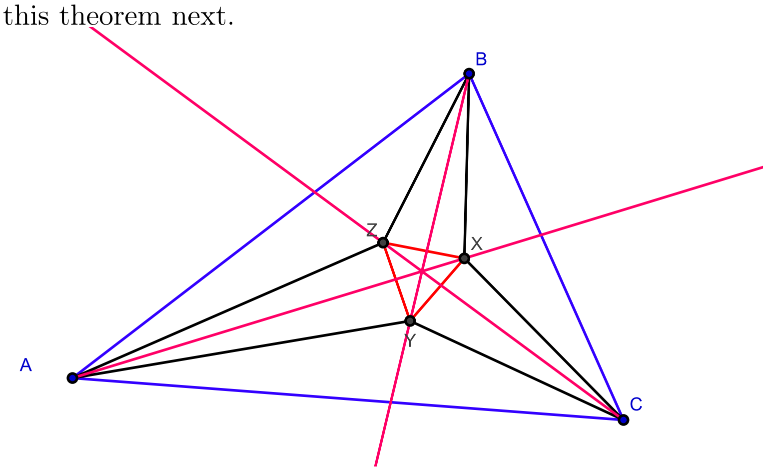
[7]In case, when vertices first triangle lie on corresponding sidelines of second triangle, this proposition isn't true.



A generalization of the Morley Point

Morley's theorem: The three points of intersection of the adjacent trisectors of the angles of any triangle form an equilateral triangle.

And one theorem says that this two triangles are perspective. And point of intersection was named $2^{nd} MorleyCentre - X(357)$[8]. We'll try to generalize this theorem next.



Let's fix $k \in [-1; 1]$. We will use the same notation in this construction like before, but with next difference:

$$\frac{\angle A_1AB}{\angle A} = \frac{\angle B_1BC}{\angle B} = \frac{\angle C_1CA}{\angle C} = k.$$

k is positive in case, when lines l_A, l_B and l_C are drawn inside and k is negative, when lines l_A, l_B and l_C are drawn outside.

[8]This notation is notation of Clark Kimberling's Encyclopedia of Triangle Centers (ETC)

It is clear that lines AX, BY and CZ intersect at a point (for $k = 1/3$ the point of intersection is $2^{nd} MorleyCentre$). Let's define the point of intersection $R(k)$. It is easy to check (maybe with Ceva's theorem) that point $R(k)$ has next barycentric coordinates:

$$R(k) = \left(\frac{sin(\angle A) \cdot sin(\angle A \cdot k)}{sin(\angle A \cdot (1-k))} : \frac{sin(\angle B) \cdot sin(\angle B \cdot k)}{sin(\angle B \cdot (1-k))} : \frac{sin(\angle C) \cdot sin(\angle C \cdot k)}{sin(\angle C \cdot (1-k))} \right)$$

It is easy to see (in this formula or in this geometrical construction) that the curve of the concurrence is self-isogonal curve. It means that $X(358)$ (isogonal conjugate of $2^{nd} MorleyCentre$) lays on this curve. Incentre lays on this curve too.

When $k = -1 : R(-1) = -\frac{1}{2}(tg\angle A : tg\angle B : tg\angle C) = H$. It means that $Circumcenter$ lays on this curve too, because $Circumcenter$ is isogonal conjugate of $Orthocenter$.

Let's find limit points:

$k = 0$: In this case

$$\frac{AL}{BL} = lim_{k \mapsto 0} \frac{sin(\angle B) \cdot sin(\angle A \cdot (1-k)) \cdot sin(\angle B \cdot k)}{sin(\angle A) \cdot sin(\angle B \cdot (1-k)) \cdot sin(\angle A \cdot k)} = \frac{sin(\angle B \cdot k)}{sin(\angle A \cdot k)} = \frac{\angle B}{\angle A}$$

Here I used that $lim_{x \mapsto 0} \frac{sinx}{x} = 1$. We have

$$R(0) = (\angle A : \angle B : \angle C)$$

$k = 1$:

$$\frac{AL}{BL} = lim_{k \mapsto 1} \frac{sin(\angle B) \cdot sin(\angle A \cdot (1-k)) \cdot sin(\angle B \cdot k)}{sin(\angle A) \cdot sin(\angle B \cdot (1-k)) \cdot sin(\angle A \cdot k)} =$$

$$= \frac{sin^2(\angle B) \cdot sin(\angle A \cdot (1-k))}{sin^2(\angle A) \cdot sin(\angle B \cdot (1-k))} = \frac{sin^2(\angle B) \cdot \angle A}{sin^2(\angle A) \cdot \angle B}$$

Here I used that $lim_{x \mapsto 0} \frac{sinx}{x} = 1$ too (like before). We have that

$$R(1) = \left(\frac{sin^2 \angle A}{\angle A} : \frac{sin^2 \angle B}{\angle B} : \frac{sin^2 \angle C}{\angle C} \right)$$

It means that $R(0)$ is isogonal conjugate of $R(1)$.

It is clear that all results in general construction (**theorem 1-4**) are true here too. Let's look at the special case of **theorem 2**. Let's define this point of intersection $D(k)$. In case when $k = 1/2$ we can receive $GergonnePoint$. Easy to find that

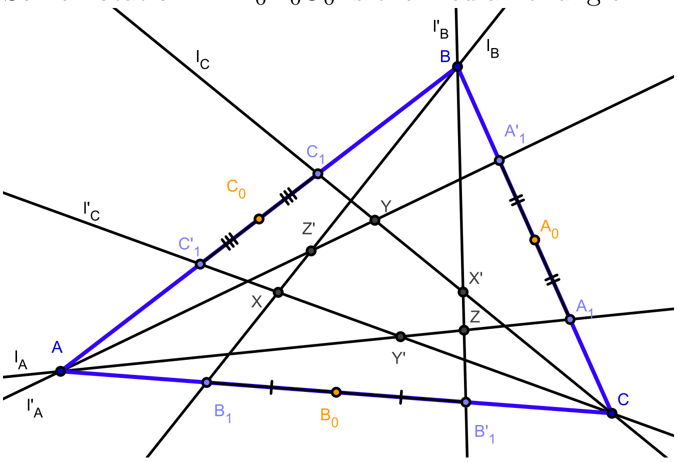
$$D(k) = (tg(\angle A \cdot k) : tg(\angle B \cdot k) : tg(\angle C \cdot k)).$$

When $k = 1$, $D(1) = H$. In a similar way we can take that $D(0) = R(0)$, like in the foregoing.

There is very difficult to find barycentric coordinates of $Q(k)$, but it is not difficult to say her limit points: $Q(0) = R(0)$ and $Q(1) = R(1)$.

Lines l_A and l'_A are isogonal lines in general design. In this construction we'll consider case where lines l_A and l'_A are isotomic lines (l_B and l'_B, l_C and l'_C are isotomic lines too next).

Some notation: $\triangle A_0B_0C_0$ is the median triangle.



Then this propositions are true:

1. Lines l_A, l'_A, l_B, l'_B, l_C and l'_C tangent the one conic. We can conclude that lines AX, BY and CZ intersect at a point (P) then. Properly lines AX', BY' and CZ' are concur too (P' is the point of intersection). And P' is isotomic conjugate of P.

2. Lines XX', YY', ZZ' and PP' are concur.

3. Points A_1, A'_1, B_1, B'_1, C_1 and C'_1 lie on the same conic section.

4. Point A_2 is point of intersection $A_1C'_1$ and $B_1A'_1$. Points B_2 and C_2 are similarly defined. Lines AA_2, BB_2 and CC_2 are concur then.

Proof.: Let's make affine transformation in which $\triangle ABC$ will translate in equilateral triangle. Thus points A_0, B_0 and C_0 will pass in the middle of a new triangle (because affine transformations save attitude of parallel pieces). Points A_1, B_1 and C_1 will pass in new points, but they will remain to lie on the corresponding edges of a new triangle. And points A'_1, B'_1 and C'_1 will pass in the points which they are symmetric to A_1, B_1 and C_1 rather the middle of the corresponding sides (besides because affine transformation save attitudes on a line). It means that the condition of theorems didn't change after transformation.

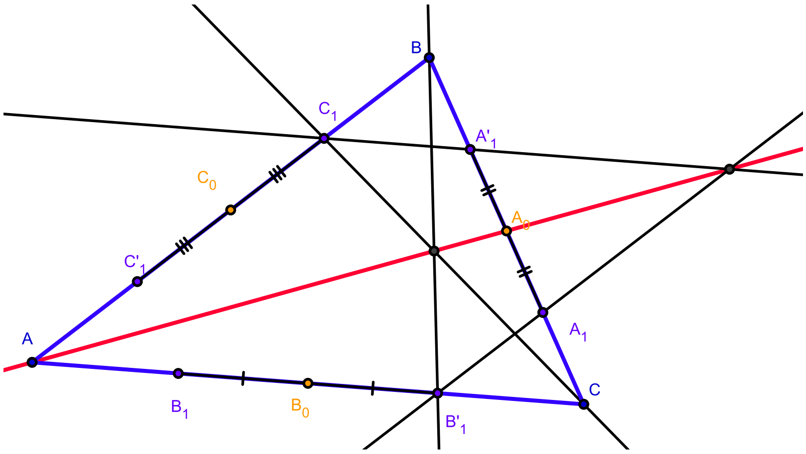
As points A_1 and A'_1 are symmetric rather A_0 and as a $\triangle ABC$ is equilateral triangle, then $\angle A_1AA_0 = \angle S'_1AA_0$. We can receive that l'_A is symmetric l_A with respect to bisecting line AA_0. Now we have construction which is analogously with general construction (but only for equilateral triangle).$\triangleleft$

Let's explore the special interesting case when pieces $A_1A'_1$, $B_1B'_1$ and

C_1C_1' are proportional to corresponding edges:

$$\frac{A_1A_1'}{BC} = \frac{B_1B_1'}{AC} = \frac{C_1C_1'}{AB}.$$

It is clear (maybe with theorem about 4 points of trapezium or with affine transformations) that points P and P' will be *Centroid* of $\triangle ABC$. Analogously, points of intersection in propositions **2** and **4** will be *Centroid* of $\triangle ABC$ too.



References

[1]A.Akopyan, A.Zaslavsky "Geometric properties of second order curves", M, MCCME, 2007.